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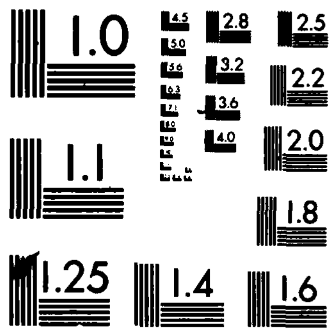

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DEVELOPMENTS IN THE THEORY OF NONLINEAR FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

The theory of scalar first-order fully nonlinear partial differential equations has recently enjoyed a strong development. One major step was a proof by M. Crandall and P.-L. Lions of the uniqueness of certain generalized solutions - called "viscosity solutions" - of problems involving such equations with the scope to accommodate applications to, for example, differential games. Following this event there has been a continuous stream of work concerning the existence, approximation and representation of viscosity solutions of Hamilton-Jacobi equations as well as the interaction of the theory of viscosity solutions and areas of application (primarily control theory and differential games), and refined uniqueness results. This survey paper, which corresponds to an invited address by the first author at an international symposium on differential equations held in March 1983 at the University of Alabama-Birmingham, introduces the relevant concepts and describes the major results up to, roughly, July 1983.

AMS (MOS) Subject Classifications: 35F20, 35F25, 35D05, 35L60, 35C05

Key Words: Hamilton-Jacobi equations, control theory, differential games, viscosity solutions

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DEVELOPMENTS IN THE THEORY OF NONLINEAR FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

Michael G. Crandall and Panagiotis E. Souganidis*

INTRODUCTION

There has been a substantial development of the theory of scalar, nonlinear, first-order partial differential equations in the last two years. It is our goal to provide a brief tour of this work and a guide to the recent literature. In view of the rate the literature is growing, this report will be somewhat out of date even on its publication, but we hope it will be useful.

In the discussion below, Ω is an open subset of R^n, $T > 0$, $\Omega_T = \Omega\times(0,T)$, "x" denotes points of R^n, u is a real-valued function of x or (t,x), $Du = (u_{x_1},\dots,u_{x_n})$ stands for the spatial gradient of u and H, ϕ, ψ are given functions of the indicated arguments. We discuss the initial-boundary value problem

$$\text{(IBVP)}\qquad \begin{aligned} &u_t + H(x,t,u,Du) = 0 \text{ for } 0 < t < T,\ x \in \Omega,\\ &u(x,t) = \phi(x,t) \text{ for } 0 < t < T,\ x \in \partial\Omega,\\ &u(x,0) = \psi(x) \text{ for } x \in \Omega, \end{aligned}$$

together with its stationary form

$$\text{(BVP)}\qquad \begin{aligned} &H(x,u,Du) = 0 \text{ for } x \in \Omega,\\ &u(x) = \phi(x) \text{ for } x \in \partial\Omega. \end{aligned}$$

A very simple case of (IBVP) arises when $\Omega = R$ and H is independent of (x,t,u). The problem then has the form

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$$(1) \qquad \begin{aligned} &u_t + H(u_x) = 0 \text{ for } x \in R,\ t > 0, \\ &u(x,0) = \psi(x) \text{ for } x \in R. \end{aligned}$$

The classical method of characteristics reveals that any smooth solution u of (1) on $\mathbb{R}\times[0,T]$ satisfies $u_x(y + tH'(\psi'(y)),t) = \psi'(y)$ for $0<t<T$ and $y \in R$. Here ψ' denotes the derivative of ψ, etc.. If the lines $x = y+tH'(\psi'(y))$ intersect for some $t > 0$ as y varies, this information is contradictory. Thus (1) cannot have smooth solutions defined for all time except in extremely special circumstances. Simple examples also show that solutions of (1) with slightly less regularity than continuous first derivatives are not unique. For example, if $H(p) = -p^2$ and $\psi = 0$, then $u \equiv 0$ and $v = \max(t - |x|,0)$ are distinct compactly supported (for bounded t) and piecewise linear solutions of (1) which satisfy the equation except on the lines $|x| = t$ and $x = 0$, where v is not differentiable.

The above remarks recall the classical observations that (1) does not have global smooth solutions in general and that the most natural weakenings of the classical notion of solution lead to nonuniqueness. However, in view of the way these problems arise in applications - in particular, in the calculus of variations, control theory and differential games - one expects a notion of solution of (IBVP) and (BVP) for which there is both existence and uniqueness.

The first demonstration of uniqueness for a notion of generalized solution of (IBVP) or (BVP) adequate to cover applications to, e.g., differential games, was given in M. G. Crandall and P. L. Lions [7]. This notion of solution is explained in Section 1 where uniqueness results are also discussed. The topic of existence is taken up in Section 2, while Section 3 is concerned with the interplay between these topics, control theory and the

theory of differential games. Approximation and representation of solutions are discussed in Section 4.

As our goal is a brief outline of recent developments, the very substantial literature which predates the work discussed herein will not be referred to. The book [28] of P. L. Lions will provide the interested reader with an appropriate historical view and references to the older literature.

There is a theory of second order equations and their relations to stochastic control and games which corresponds to the first order theory reviewed in this paper. We have not discussed this theory herein and refer the interested reader to the papers [29], [30] of P. L. Lions in this regard. The topic of quasi-variational inequalities and Hamilton-Jacobi equations, which is not otherwise mentioned herein, is taken up in [1].

SECTION 1. NOTIONS OF SOLUTION AND UNIQUENESS

It will be convenient to consider a general equation of the form

$$F(y,u,Du) = 0 \text{ in } O, \tag{1.1}$$

where $O \subseteq \mathbb{R}^m$ is open, $y = (y_1, y_2, \ldots, y_m)$, $Du = (u_{y_1}, \ldots, u_{y_m})$, and $F: O \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$. Of course, this general form incorporates the equations in both (IBVP) and (BVP). For $u \in C(O)$ and $z \in O$ put

$$\begin{aligned} D^+u(z) &= \left\{ p \in \mathbb{R}^m : \limsup_{y \to z} \frac{(u(y) - (u(z) + p\cdot(y-z)))}{|y-z|} \le 0 \right\}, \\ D^-u(z) &= \left\{ p \in \mathbb{R}^m : \liminf_{y \to z} \frac{(u(y) - (u(z) + p\cdot(y-z)))}{|y-z|} \ge 0 \right\}, \end{aligned} \tag{1.2}$$

where $a \cdot b$ is the scalar product of $a, b \in \mathbb{R}^m$. E. g., if $O = (-1,1)$ and $u(y) = |y|$, then $D^+u(0) = \emptyset$ and $D^-u(0) = [-1,1]$. The relation $p \in D^+u(z)$ can be written $u(y) \le u(z) + p\cdot(y - z) + o(y - z)$, with the usual meaning of $o(y-z)$, and a similar remark applies to D^-. Clearly u is differentiable at

$z \in \mathcal{O}$ exactly when both $D^+u(z)$ and $D^-u(z)$ are both nonempty and then

$$D^+u(z) = D^-u(z) = \{ Du(z) \}$$

where $Du(z)$ denotes the usual (Fréchet) derivative of u at z. Since there are continuous functions u which are nowhere differentiable, there are continuous functions u such that at least one of $D^+u(z)$ and $D^-u(z)$ is empty at every point $z \in \mathcal{O}$. One way to define a <u>viscosity solution</u> of (1.1) is:

<u>Definition 1.</u> Let $u \in C(\mathcal{O})$. Then u is a viscosity solution of $F < 0$ in $\mathcal{O}$ if

(1.3) $$F(y,u(y),p) < 0 \text{ for all } y \in \mathcal{O} \text{ and } p \in D^+u(y).$$

Similarly, u is a viscosity solution of $0 < F$ in $\mathcal{O}$ if

(1.4) $$0 < F(y,u(y),p) \text{ for all } y \in \mathcal{O} \text{ and } p \in D^-u(y).$$

Finally, u is a viscosity solution of $F = 0$ in $\mathcal{O}$ if it is a viscosity solution of both $F < 0$ and $0 < F$.

We will also call a viscosity solution of $F < 0$ (respectively, $0 < F$) a viscosity subsolution (supersolution) of $F = 0$. The obvious relations between D, D^+, D^- guarantee that the notion of a viscosity solution is consistent with that of a classical solution. For example, continuously differentiable solutions of $F = 0$ in the classical sense are viscosity solutions and a viscosity solution u of $F = 0$ satisfies $F(y,u(y),Du(y)) = 0$ at each point $y \in \mathcal{O}$ at which u is differentiable.

The term "viscosity solution" arises because a standard method to approach the existence of solutions of, e.g., (IBVP) is to solve (if possible)

(1.5) $$u_{\varepsilon t} + H(x,t,u_\varepsilon,Du_\varepsilon) - \varepsilon\Delta u_\varepsilon = 0,$$

where Δ denotes the Laplacian in R^n, subject to initial and boundary conditions and then to show that u_ε has a limit point in $C(\mathcal{O})$ as $\varepsilon \downarrow 0$ with the desired properties. This is the method of "vanishing viscosity".

Here are two other ways to define a viscosity solution.

Definition 2. A viscosity solution of $F = 0$ is a $u \in C(O)$ such that if $\phi \in C_0^\infty(O)$ is nonnegative and $k \in R$, then (i) and (ii) below hold:

(i) If $\phi(u - k)$ has a positive maximum in O, then

$$F\left(y,u(y),\frac{-D\phi(y)(u(y) - k)}{\phi(y)}\right) \le 0 \tag{1.6}$$

at some point y at which the maximum is achieved.

(ii) If $\phi(u - k)$ has a negative minimum in O, then (1.6) holds with the opposite inequality at some point y at which the minimum is achieved.

Definition 3. A viscosity solution of $F = 0$ is a $u \in C(O)$ such that for every $\phi \in C^1(O)$, $F(y,u(y),D\phi(y)) \le 0$ (respectively, $0 \le F(y,u(y),D\phi(y))$) holds at each relative maximum (respectively, minimum) point $y \in O$ of $u - \phi$.

The equivalence of these three definitions is proved in [7] (in which Definition 2 was taken as the basic property). Definition 3 is related to ideas used by L. C. Evans in [10]. Definition 2 is obviously convenient for localization and for this reason uniqueness was first obtained using it. Subsequently, in [6], a variety of things were proved using Definitions 1 and 3 directly. We recommend [6] for an introduction to properties of viscosity solutions and for easy access to proofs of model cases of results like those given below.

Let us formulate conditions on H under which one can prove uniqueness results. B_R will denote the R-ball in R^n. The first two conditions are the basic continuity requirement and a monotonicity condition concerning the dependence of H on u. (For those who may wonder, it is the monotonicity condition that rules out the case of a conservation law in what follows).

$$\text{(H1)}\qquad \begin{array}{l} H:\bar{\Omega}\times[0,T]\times R\times R^n \to R \text{ is uniformly continuous} \\ \text{on } \Omega\times[0,T]\times[-R,R]\times B_R \text{ for each } R > 0. \end{array}$$

For each $R > 0$ there is a constant $\gamma_R > 0$ such that

(H2) $$H(x,t,r,p) - H(x,t,s,p) \geq \gamma_R(r - s)$$

for $x \in \Omega$, $p \in \mathbb{R}^n$ and $-R \leq s \leq R$.

The next two hypotheses concern the behavior of the uniform continuity in x as p becomes large. They are expressed in terms of the function

$$\Lambda(R,\rho,\varepsilon) = \sup\{|H(x,t,r,p)-H(y,t,r,p)|:0\leq t\leq T, |r|\leq R, |x-y||p|\leq\rho, |x-y|\leq\varepsilon\}.$$

Here $| \; |$ denotes both the norm in $\mathbb{R}^n$ and in $\mathbb{R}$. The weaker of the two hypotheses is

(H3) $$\lim_{\varepsilon\downarrow 0} \Lambda(R,\varepsilon,\varepsilon) = 0 \text{ for } R > 0.$$

Since Λ is nondecreasing in its arguments a more severe restriction is

(H4) $$\lim_{\varepsilon\downarrow 0} \Lambda(R,\rho,\varepsilon) = 0 \text{ for } R,\ \rho > 0.$$

The conditions (H1) - (H4) are meaningful for the problem (BVP) as well when H is interpreted as a function of t which happens to be independent of t. The uniqueness result of [7] for (BVP) is:

Theorem ([7]). Let u, v be bounded continuous functions on $\bar{\Omega}$ which are viscosity solutions of $H = 0$ in Ω. Let ϕ be continuous on $\partial\Omega$ and $u(x) - \phi(z)$, and $v(x) - \phi(z)$ tend to zero as $x \to z \in \partial\Omega$ uniformly in z. Set

$$R = \max\left(\|u\|_{L^\infty(\Omega)}, \|v\|_{L^\infty(\Omega)}\right)$$

and let (H1) and (H2) hold with $\gamma_R > 0$ in (H2). Then:

(i) If (H4) holds, then $u = v$.

(ii) If u and v are uniformly continuous and (H3) holds, then $u = v$.

(iii) If u and v are Lipschitz continuous, then $u = v$.

This result in fact follows from more general estimates comparing viscosity sub-and-super solutions of different problems. We will not formulate these results here. Observe that as the hypotheses on u and v are strengthened, less is required of H. The corresponding result for (IBVP) is

quite similar. The statement for this case arises upon replacing Ω by $\Omega\times[0,T]$, $\partial\Omega$ by the union of $\partial\Omega\times[0,T]$ and $\Omega\times\{0\}$, and $\phi(x)$ by $\phi(x,t)$ on $\partial\Omega\times[0,T]$ and by $u_0(x)$ when $t = 0$. Moreover, the requirement $\gamma_R > 0$ in (H2) is dropped. (The linear function $\gamma_R(r - s)$ in (H2) is replaced by a nonlinear function in the result for (BVP) in [7].) The necessity of conditions like (H3) or (H4) is shown via examples in [7].

Before the results mentioned above, the main uniqueness results which were established in a generality for which there was a corresponding global existence theory concerned the case of convex Hamiltonians. These results concern solutions of the equations in an almost everywhere sense which also satisfy a type of "semi-concavity" condition. Concerning such results we refer to the book of P. L. Lions [28]. The viscosity notion is used in [28], but the main emphasis in this book is the important special case of viscosity solutions which are Lipschitz continuous (and hence satisfy the equation almost everywhere).

Other uniqueness results concern domains of dependence (e.g., [7]), unbounded functions, (H. Ishii [21]) and Hamiltonians which are not necessarily continuous in t (H. Ishii [20]). With respect to domains of dependence, observe that if we regard (IBVP) as a special case of (BVP) by thinking of t as as "space variable", then we have not prescribed data at $t = T$. General results concerning which part of the boundary of Ω is important for uniqueness in (BVP) are the subject of work of R. Jensen ([22]) in progress at the time of this symposium.

SECTION 2. EXISTENCE

The existence theory for viscosity solutions of (IBVP) and (BVP) is much more a continuation of the existence theory which predates the notion of viscosity solutions than the corresponding uniqueness theory (which is quite distinct from what existed before) is a continuation of more classical results. Roughly speaking, known methods adapt to proving the existence of viscosity solutions and the flexibility of the notion allows one to take limits freely and obtain new results. There are also new arguments which arose partly in trying to get the existence theory in harmony with the generality of the uniqueness theory. We are going to describe, in more or less chronological order, results obtained since the introduction of viscosity solutions and ask the reader to be aware that this does not give an accurate historical view. We again refer to [28] for a more balanced view of the earlier theory.

The sort of dramatic existence and uniqueness theorems which are now possible may be illustrated by the model problems

$$u + H(Du) = v \text{ in } \mathbb{R}^n, \tag{2.1}$$

and

$$\begin{aligned} &u_t + H(Du) = 0 \text{ in } \mathbb{R}^n \times (0,T], \\ &u(x,0) = u_0(x) \text{ in } \mathbb{R}^n. \end{aligned} \tag{2.2}$$

It was proven in [7] that if H is continuous from $\mathbb{R}^n$ to $\mathbb{R}$ and v and u_0 are bounded and uniformly continuous, then (2.1) and (2.2) have viscosity solutions to which the uniqueness theorem applies. The only regularity required is continuity of H and uniform continuity of v and u_0, and then there is a unique global solution.

P. L. Lions, in [27] and [28], considers problems of the forms (IBVP) and (BVP). Two types of assumptions on H are important in his work. One is a

continuity assumption of Lipschitz type which is used to establish uniform Lipschitz estimates on solutions of (IBVP) and (BVP) (using results of [26]). A simple special case of this hypothesis reads

For $R > 0$ there is a constant C_R such that

(H5) $$|H(t,x,r,p) - H(t,y,r,p)| \le C_R(1 + |p|)|x - y|$$

for $t \in [0,T]$, x, $y \in \bar{\Omega}$, $p \in R^n$ and $|r| \le R$.

The other assumption reads

(H6) $H(t,x,r,p) \to \infty$ as $|p| \to \infty$, uniformly for $x \in \bar{\Omega}$, $0 \le t \le T$, and $r \in R$.

The conclusion - in either case - is the exi· .n·e of bounded, Lipschitz continuous viscosity solutions provided the .ندary conditions are suitable (see below). The proofs in [28] are based . ‵he vanishing viscosity method, while in [27] a method of penilization of domain (introduced in [29]) is employed.

The paper [33] deals with (IBVP) and the "resolvent-type"stationary problem

$$u + \lambda H(x,u,Du) = v \text{ for } \lambda > 0$$

in $R^n \times [0,T]$ and R^n respectively. The result is the existence of bounded and uniformly continuous viscosity solutions under hypotheses (H4) or (H5), where in the latter case the dependence of H on (t,r,p) is less restrictive that that used in [27], [28]. In both cases the vanishing viscosity method is used and the proof proceeds via estimating the modulus of continuity of solutions of the approximate problems with respect to x.

However, the best theorem concerning existence of bounded uniformly continuous viscosity solutions (without (H6)) was obtained by G. Barles [1]. Barles was able to show existence under the assumption (H3), which is the weakest assumption under which uniqueness of bounded uniformly continuous solutions has been proved. The notation BUC(K) will mean the bounded and

uniformly continuous functions on K. Barles provided the existence assertion of the next theorem which is stated for the general case in which Ω is not $\mathbb{R}^n$, but it is valid equally well when Ω is $\mathbb{R}^n$ (there simply being no conditions on $\partial\Omega$ in this event).

Theorem ([1]).

(a) Let $H:\bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ satisfy (H1), (H2) and (H3) with $\gamma_R > 0$ for $R > 0$. Let $\underline{u}$, $\overline{u} \in BUC(\bar{\Omega})$ be, respectively, a sub- and a supersolution of $H = 0$ in Ω. Assume, moreover, that $\underline{u} = \overline{u}$ on $\partial\Omega$. Then there is a unique viscosity solution $u \in BUC(\bar{\Omega})$ of $H = 0$ in Ω such that $u = \underline{u} = \overline{u}$ on $\partial\Omega$.

(b) Assume that $H:\bar{\Omega}\times[0,T]\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ satisfies (H1), (H2), and (H3). Let $\underline{u}$, $\overline{u} \in BUC(\Omega\times[0,T])$ be, respectively, a sub- and a supersolution of $u_t + H(x,t,u,Du) = 0$ in $\Omega\times[0,T]$ such that $\underline{u}(x,t) = \overline{u}(x,t) = \phi(x,t)$ for (x,t) in $\partial\Omega\times[0,T]$ and $\underline{u}(x,0) = \overline{u}(x,0) = \psi(x)$ in Ω. Then there is a unique viscosity solution $u \in BUC(\bar{\Omega}\times[0,T])$ of (IBVP).

Barles attacked the existence problem by reducing the question to one of the form $v + \bar{H}(y,v,Dv) = 0$ in $\mathbb{R}^m$ for some m and then approximating the Hamiltonian $\bar{H}$ involved by smooth ones satisfying an (H5)-type assumption. The results of [27] together with the uniqueness estimates of [7] are used to complete the proof.

Of course, given the above theorem, interest shifts to the question of when sub - and supersolutions with the desired properties can be found. There are many open questions here. Especially when Ω is not $\mathbb{R}^n$, it is only in the case of convex Hamiltonians that this is well understood, and one can find necessary and sufficient conditions in [28].

Recently, L. C. Evans and H. Ishii [13] gave a different proof for the existence of a viscosity solution of $u + H(x,Du) = 0$ in Ω, $u = \phi$ on $\partial\Omega$, where

H satisfies (H3). Their technique involves a version of the localization method of [29].

SECTION 3. CALCULUS OF VARIATIONS, OPTIMAL CONTROL THEORY AND DIFFERENTIAL GAMES.

Hamilton-Jacobi equations are classically derived in the closely related areas of the calculus of variations, optimal control theory (where they are called Bellman equations) and differential games (Isaacs equations). As was observed by Bellman, a very good way to exhibit the relationship between the equations and the problems in which they arise is via the dynamic programming principle (see, for example, Fleming and Rishel [17]). This remains true in the context of viscosity solutions as was pointed out by P. L. Lions. In this section we will expand a bit on these remarks and give several references.

We begin by reviewing the relationship between the notion of a viscosity solution and control theory in the simplest possible case: We consider a "finite horizon" control problem without boundary conditions and formulate two typical theorems (omitting all the technical assumptions - the references contain precise statements). After this we quickly outline the situation as regards differential games. Consider a system whose state at time t if started at x is $y_x(t)$ and whose evolution is governed by the following Cauchy problem for an ordinary differential equation:

$$\begin{aligned} &\dot{y}_x(t) + f(y_x(t),v(t)) = 0 \text{ for } 0 < t, \\ &y_x(0) = x \in \mathbb{R}^N, \end{aligned} \tag{3.1}$$

where $f:\mathbb{R}^N \times V \to \mathbb{R}^N$, V is a given compact set in some $\mathbb{R}^M$ (for example) and v(t), called the control, is a measurable function from $[0,\infty)$ to V. Under natural assumptions on f, (3.1) has a unique solution for all $x \in \mathbb{R}^N$ denoted by $y_x(t)$. We now define a pay-off $J(x,t;v(\cdot))$ for each given x,t and control $v(\cdot)$:

$$(3.2) \qquad J(x,t;v(\cdot)) = u_0(y_x(t)) + \int_0^t h(y_x(s),v(s))ds,$$

Here $h(x,v)$ and u_0 are given continuous functions which satisfy certain hypotheses. The problem is to choose the control function $v(\cdot)$ so as to minimize J. An important quantity in this situation is the <u>value</u> (or the <u>optimal cost</u>):

$$(3.3) \qquad u(x,t) = \inf_{v(\cdot)} J(x,t;v(\cdot)).$$

Two of the main goals of control theory are to characterize u and to design optimal (or nearly optimal) controls. One of the most important tools for the study of these problems is the dynamic programming principle. In the case under consideration here it says:

<u>Theorem (dynamic programming).</u> For all $0 < s < t$

$$u(x,t) = \inf_{v(\cdot)} \left\{ \int_0^s h(y_x(\sigma),v(\sigma))d\sigma + u(y_x(s),t-s)\right\}.$$

Using this principle P.L. Lions proved [28]:

<u>Theorem.</u> The optimal cost u given by (3.3) is the viscosity solution of

$$u_t + \sup_{v \in V} \{f(x,v)\cdot Du - h(x,v)\} = 0 \text{ in } R^n \times (0,\infty),$$

$$u(x,0) = u_0(x) \text{ in } R^n.$$

Observe that the Hamiltonian here, that is

$$H(x,p) = \sup\{f(x,v)\cdot p - h(x,v): v \in V\},$$

is the supremum of affine functions and is therefore convex. For the proofs and generalizations of these results and references to the classical literature, see P. L. Lions [28], [24], and L. C. Evans [12]. Other recent references which are relevant for the interaction between viscosity solutions and contol theory are: P. L. Lions and M. Nisio [31], I. Capuzzo Dolcetta and

L. C. Evans [4], L. C. Evans and H. Ishii [13], I. Capuzzo Dolcetta [3], I. Capuzzo Dolcetta and H. Ishii [5].

Differential games resemble contol theory in involving a system of differential equations and a pay-off function. However, now there are two players and two control sets. The first player attempts to make the pay-off large by choosing his control, while the second player strives to keep the pay-off small by judicious choice of his control. The notion of value becomes much more subtle because, as the words suggest, we are dealing with "sup-inf" or "inf-sup" operations (as opposed to the simple "sup" of the control case) depending on which player "goes first" and this must be interpreted in some infinitesimal way. However, the value may be defined in various ways which turn out to be equivalent - and a very good way to see the equivalence is to use the relation with Hamilton-Jacobi equations and the uniqueness theory. The formal relationships between differential games and Hamilton-Jacobi equations were observed by Isaacs [19] in the early 1950's: he showed that if the values of various differential games were regular enough, then they solve certain first-order partial differential equations (called Isaacs equations) with "max-min" or "min-max" type nonlinearities (in the sense that the Hamiltonian for the control problem is a "max" nonlinearity). The difficulty that these equations seldom have solutions regular enough to justify the formal arguments was worked around in various ways by, among others, W. Fleming [15], [16], A. Friedman [18], Elliot and Kalton [9] (see also the references in these works). A main idea here was to approximate the differential game by a stochastic differential game whose value would then satisfy a vanishing - viscosity type of approximation of the first-order equation. It proved possible to take a unique limit as the noise went to 0 and to identify this limit with the value. Again, the unique existence of

this limit - which was relatively difficult to show - is made transparent by uniqueness results like those of Section 1.

For direct proofs of the facts that (various notions of) the value of differential games satisfy the Isaac's equation in the viscosity sense as well as some applications we refer to P. Souganidis [32], E. Barron, L. C. Evans and R. Jensen [2], L. C. Evans and H. Ishii [13], L. C. Evans and P. Souganidis [14], H. Ishii [20], and P. L. Lions [24].

SECTION 4. REPRESENTATION AND APPROXIMATION OF SOLUTIONS

It will be advantageous to sketch the topics of this section in the context of the model problem

$$(4.1)\qquad \begin{aligned} &u_t + H(Du) = 0 \text{ for } t > 0,\ x \in \mathbb{R}^n, \\ &u(x,0) = \psi(x), \text{ for } x \in \mathbb{R}^n, \end{aligned}$$

where $H:\mathbb{R}^n \to \mathbb{R}$ is continuous. The proof of the uniqueness results of Section 1 and the existence results of Section 1 and 2 provide a unique bounded viscosity solution u of (4.1) which is uniformly continuous on $\mathbb{R}^n \times [0,T]$ for each $\psi \in BUC(\mathbb{R}^n)$ and $T > 0$. Moreover, the self-map $S(t)$ of $BUC(\mathbb{R}^n)$ given by $S(t)\psi(x) = u(x,t)$ where u is the solution of (4.1) is a semigroup (i.e., $S(t)S(s) = S(t + s)$) which preserves the natural order of $BUC(\mathbb{R}^n)$ and satisfies

$$\|S(t)\psi - S(t)\eta\| \le \|\psi - \eta\| \text{ for } 0 \le t \text{ and } \psi, \eta \in BUC(\mathbb{R}^n).$$

Here $\|\ \|$ denotes the essential supremum norm. We refer to this norm estimate by saying simply that S(t) is "nonexpansive". We first formulate an abstract theorem about approximating S(t) and then give some special cases of it.

For $\rho > 0$, let $F(\rho):BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ satisfy:

(F1) $F(\rho)$ is and order-preserving, translation invariant and nonexpansive self-map of $BUC(\mathbb{R}^n)$.

(F2) For each $\phi \in BUC(\mathbb{R}^n)$, $\rho \to F(\rho)\phi$ is continuous from $[0,\infty)$ into $BUC(\mathbb{R}^n)$.

(F3) There is a constant C such that

$$\left\|\frac{F(\rho)\phi - \phi}{\rho} + H(D\phi)\right\| \leq C(1 + \|D\phi\| + \|D^2\phi\|)\rho$$

for $\rho > 0$ and those $\phi \in BUC(\mathbb{R}^n)$ whose first and second partial derivatives are continuous and bounded on $\mathbb{R}^n$.

In (F3), $\|D^2\phi\|$ denotes the largest supremum norm of any second derivative of ϕ, etc.. We will approximate S by the following object built from F:

(4.2) $S_\lambda(t)\psi = F(t - n\lambda)F(\lambda)^n\psi$ if $n\lambda \leq t \leq (n+1)\lambda$, $n = 0,1,2,.$, and $\lambda > 0$.

<u>Theorem ([32]).</u> let F satisfy (F1), (F2), (F3), $\psi \in BUC(\mathbb{R}^n)$ and $T > 0$. Then

(4.3) $$\lim_{\lambda\to 0} \|S(t)\psi - S_\lambda(t)\psi\| = 0 \text{ uniformly for } 0 \leq t \leq T.$$

If also H and ψ are Lipschitz continuous, then there is a constant K such that

(4.4) $$\|S(t)\psi - S_\lambda(t)\psi\| \leq K\sqrt{\lambda} \text{ for } 0 \leq t \leq T.$$

We remark that the validity of (4.3) only requires that $(F(\rho)\phi - \phi)/\rho \to -H(D\phi)$ uniformly when ϕ is bounded in C^2. The strong estimate (4.4) requires the stronger criteria in (F3).

Let us give some examples.

<u>Min-max representations:</u> Assume that Y and Z are compact subsets of some $\mathbb{R}^m$ and $h:Y\times Z \to \mathbb{R}$, $f:Y\times Z \to \mathbb{R}^n$ are bounded. Let H be given by

$$H(p) = \inf_{y\in Y} \sup_{z\in Z} (h(y,z) + f(y,z)\cdot p).$$

If we set

$$F(\rho)\psi(x) = \sup_{y\in Y}\inf_{z\in Z} (-\rho h(y,z) + \psi(x - \rho f(y,z))),$$

then (F1) - (F3) hold, as a straightforward analysis shows. The theorem, in this case, represents $S(t)\psi$ as the limit of iterated "max-min" operations and provides an error estimate.

A difference approximation:

Here we take $N = 1$ to simplify the writing. The equation becomes $u_t + H(u_x) = 0$. Assume that H is Lipschitz continuous and consider the difference approximation

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + H\left(\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} \right) - \Theta\Delta x \frac{U_{j+1}^n + U_{j-1}^n - 2U_j^n}{(\Delta x)^2} = 0$$

of this equation, in which Θ is a constant to be chosen, Δx and Δt are step sizes and U_j^n corresponds to $u(j\Delta x, n\Delta t)$. This approximation is consistent with the equation and involves the viscosity term $-(\Theta\Delta x)(U_{j+1}^n + U_{j-1}^n - 2U_j^n)$, which corresponds roughly to $-\Theta(\Delta x)u_{xx}$. Let $\alpha = \Delta t/\Delta x$. An "F" corresponding to this aproximation is

$$(4.5) \quad F(\rho)\psi(x) = \psi(x) - \alpha\rho H\left(\frac{\psi(x+\rho) - \psi(x-\rho)}{2\rho} \right) + \lambda\Theta(\psi(x+\rho) + \psi(x-\rho) - 2\psi(x)).$$

This F has the desired properties (F1) - (F3) if $2\alpha\Theta < 1$ and $L < \Theta$, where L is the Lipschitz constant for H. The theorem implies that if $2\alpha\Theta < 1$ and $L < \Theta$, ψ is Lipschitz continuous and $U_j^0 = \psi(j\Delta x)$, then the U_j^n given by (4.5) for $n = 1,2,\ldots$ and all j satisfy

$$|U_j^n - S(n\Delta t)\psi(j\Delta x)| < C\sqrt{(\Delta t)}$$

for a suitable C. (We leave it to the reader to trace this relation.)

A product formula:

Let the Hamiltonian H in (4.1) be given as a sum: $H(p) = H_1(p) + H_2(p)$ where each of H, H_1, and H_2 is Lipschitz continuous. Let $S(t)$, $S_1(t)$, and

$S_2(t)$ be the semigroups associated with the problem (4.1) for the Hamiltonians H, H_1, and H_2 respectively. Put

$$F(\rho) = S_1(\rho)S_2(\rho).$$

It is again possible to verify (F1) - (F3), although it is not so easy as in the previous two cases. As a consequence, one has

$$\|S(t)\psi - (S_1(t/n)S_2(t/n))^n\psi\| \le C/\sqrt{n}$$

for $n = 1,2,\ldots$ and bounded $0 \le t$.

With these examples to set the scene, we outline the current situation. Approximation of viscosity solutions by a max-min representation was first established by L. C. Evans [11]. The abstract theorem is a very special case of results of Souganidis [32], wherein one finds more general applications to max-min representations. These are closely related to Fleming's notion of the value of a differential game (see [32]). The Elliot - Kalton notion yields a different representation theorem ([13], [14]), as does the Friedman notion ([32], [2]). A different max-min representation (where the index sets are finite) is given by Lenhart [23].

The first error estimates like (4.4) were given in [8] in the case of numerical approximations. A general class of difference approximations of (4.1) are discussed in [32] and error estimates are made. Moreover, error estimates are made in the same spirit for the approximation of (4.1) by (1.5). These reproduce in a much simpler way estimates inherent in the older literature.

The proofs of the abstract results of [32] are related to those of [8]. The application of the general convergence results in [32] to numerical approximations extend the results of [8] to more general Hamiltonians, etc.. Implicit schemes are also considered. Product formulae related to the example above are also found in [32].

Other works of interest include [5], in which a stationary problem with a convex Hamiltonian is approximated by a "min" method and the convexity permits a sharper error estimate.

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The theory of scalar first-order fully nonlinear partial differential equations has recently enjoyed a strong development. One major step was a proof by M. Crandall and P.-L. Lions of the uniqueness of certain generalized solutions - called "viscosity solutions" - of problems involving such equations with the scope to accommodate applications to, for example,
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ABSTRACT (cont.)

differential games. Following this event there has been a continuous stream of work concerning the existence, approximation and representation of viscosity solutions of Hamilton-Jacobi equations as well as the interaction of the theory of viscosity solutions and areas of application (primarily control theory and differential games), and refined uniqueness results. This survey paper, which corresponds to an invited address by the first author at an international symposium on differential equations held in March 1983 at the University of Alabama-Birmingham, introduces the relevant concepts and describes the major results up to, roughly, July 1983.

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